PASCAL'S WAGER AND THE PARADOX OF KRAITCHIK

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Recently, I was looking at the famous Pascal's Wager.

Pascal lived from 1623 to 1662. He was renown as a French mathematician, physicist and philosopher. I invented the first calculator.

I give it here, first the original text (in old French), and a translation (I think "not too bad, not too good") I could find:

Original text:

Examinons donc ce point, et disons : Dieu est ou il n'est pas ; mais de quel côté pencherons-nous ? La raison n'y peut rien déterminer. Il y a un chaos infini qui nous sépare. Il se joue un jeu à l'extrémité de cette distance infinie, où il arrivera croix ou pile. Que gagerez-vous ? Par raison, vous ne pouvez faire ni l'un ni l'autre ; par raison, vous ne pouvez défendre nul des deux.

Ne blâmez donc pas de fausseté ceux qui ont pris un choix, car vous n'en savez rien. - Non, mais je les blâmerai d'avoir fait non ce choix, mais un choix, car encore que celui qui prend croix et l'autre soient en pareille faute, il sont tous deux en faute ; le juste est de ne point parier.

- Oui, mais il faut parier. Cela n'est point volontaire, vous êtes embarqué. Lequel prendrez-vous donc ? Voyons, puisqu'il faut choisir, voyons ce qui vous intéresse le moins. Vous avez deux choses à perdre, le vrai et le bien, et deux choses à engager, votre raison et votre volonté, votre connaissance et votre béatitude, et votre nature a deux choses à fuir, l'erreur et la misère. Votre raison n'est pas plus blessée, puisqu'il faut nécessairement choisir, en choisissant l'un que l'autre. Voilà un point vidé. Mais votre béatitude ? Pesons le gain et la perte en prenant croix que Dieu est. Estimons ces deux cas : si vous gagnez, vous gagnez tout, et si vous perdez, vous ne perdez rien ; gagez donc qu'il est sans hésiter. Cela est admirable.

Mais je gage peut-être trop. Voyons : puis qu'il y a pareil hasard de gain et de perte, quand vous n'auriez que deux vies à gagner pour une, vous pourriez encore gager. Et s'il y en avait dix à gagner, vous seriez bien imprudent de ne pas hasarder votre vie pour en gagner dix à un jeu où il y a pareil hasard de perte et de gain.
Mais il y a ici une infinité de vies infiniment heureuses à gagner avec pareil hasard de perte et de gain ; et ce que vous jouer est si peu de chose, et de si peu de durée, qu'il y a de la folie à le ménager en cette occasion.

Translation :

"God is, or He is not." But to which side shall we incline? Reason can decide nothing here. There is an infinite chaos which separated us. A game is being played at the extremity of this infinite distance where heads or tails will turn up... Which will you choose then? Let us see. Since you must choose, let us see which interests you least. You have two things to lose, the true and the good; and two things to stake, your reason and your will, you knowledge and your happiness; and your nature has two things to shun, error and misery. Your reason is no more shocked in choosing one rather than the other, since you must of necessity choose... But your happiness? Let

us weigh the gain and the loss in wagering that God is... If you gain, you gain all; if you lose, you lose nothing. Wager, then, without hesitation that He is.

That is very fine. Yes, I must wager; but I may perhaps wager too much.

Let us see. Since there is an equal risk of gain and of loss, if you had only to gain two lives, instead of one, you might still wager. But if there were three or even ten lives to gain, you would have to play (since you are under the necessity of playing), and you would be imprudent, when you are forced to play, not to chance your life to gain three or even ten at a game where there is an equal risk of loss and gain. But there is an eternity of life and happiness.

When you look on Internet, for instance with Google, using “pascal’s wager” or “pascal wager”, you find more than 10 000 articles,a lot from logicians who have tried to see what’s can be wrong in this wager.

The interesting fact is that – for what I have see – none of them made any comparison of Pascal’s Wager and Kraïtchik’s Paradox!

Here, I have to present Kraïtchik’s Paradox:

More than twenty years ago, I read the book “La mathématique des jeux” of Maurice Kraitchik. (First edition: Imprimerie Stevens, Bruxelles, 1930; Second edition - which I have -: Editions techniques et scientifiques, Bruxelles, 1953).

It’s a fascinating book, with a lot of mathematical puzzles, considerations on magic squares, geometrical curiosities, …

Who was Maurice Kraitchik (1882 – 1957)?

He was a Belgian mathematician (born in Russia) whose primary interests were the theory of numbers and recreational mathematics, on both subjects of which he published a lot. He wrote several books on number theory (1922-1930, and after the war), and was the editor of the periodical *Sphinx* (1931-1939), which was devoted to recreational mathematics. During World War II, Kraïtchik emigrated to the United States, where he taught a course at the New School for Social Research in New York City on the general topic of "mathematical recreations." Kraïtchik was « agrégé » of the free University of Brussels, engineer at the “Société Financière de Transports et d'Entreprises Industrielles (Sofina)”, and director of the “Institut des Hautes Etudes de Belgique”.

Among his books, let’s mention:

Kraïtchik, M. *Théorie des Nombres.* Paris: Gauthier-Villars, 1922.
Kraïtchik, M. *Recherches sur la théorie des nombres.* Paris: Gauthier-Villars, 1924.
Kraïtchik, M. *Mathematical Recreations.* New York: Dover, 1953.
Kraïtchik, M. *Alignment Charts.* New York: Van Nostrand, 1944.

In “La mathématique des jeux”, I considered during years one of the paradox he presents (page 133): “Deux personnes, également « riches » conviennent de comparer les contenus de leurs porte-monnaies. Chacun ignore les contenus des deux porte-monnaies. Le jeu consiste en ceci : Celui qui a le moins d’argent reçoit le contenu du porte-monnaie de l'autre. (au cas où les montants sont égaux, il ne se passe rien). Un des deux hommes peut penser : « Admettons que j'ai un montant de A$ dans mon porte-monnaie. C'est le maximum que je peux perdre. Si je gagne (probabilité 0.5), le montant final en ma possession sera supérieur à 2A. Donc le jeu m'est favorable… l'autre homme fait exactement le même raisonnement. Bien entendu, vu la symétrie, le jeu est équilibré. Où est la faute dans le raisonnement de chaque homme ? »

Two people, equally "rich" put their wallets on the table. Both don't know the amounts of money of each wallet. The game is : "the man who has the less money receives the money from the other" (if they have the same amount, nothing happens). One of the men may think : "I know I have an amount of A$ in my wallet. That's the maximum I can lose. If I win (probability 0.5), my final amount of money will be greater than 2A. So the game is in my favor"...the other man thinks exactly the same. Of course, because of symmetry, the game is equilibrated. What is wrong with the reasoning of the two men?

I noticed that Martin Gardner, in "La magie des paradoxes" (Bibliothèque POUR LA SCIENCE - Diffusion Belin, extracts of Scientific American, 1975), page 114, gives the same problem, asking for an answer (« I was not able to solve it »).

Martin Gardner (born in 1914) was the Mathematical Games columnist for Scientific American. He originated the column in 1956, and his columns appeared until his retirement from the magazine in 1986. He graduated Phi Beta Kappa from the University of Chicago in 1936.
In her book "The power of logical thinking" (St. Martin's Griffin Edition, 1997), Marilyn Vos Savant mentions Martin Gardner as a very logical thinker.

Some of his mathematical titles (published by several editors):
The Scientific American Book of Mathematical Puzzles and Diversions.
The Magic Numbers of Dr. Matrix.
Fractal Music, Hypercards and More.
Codes, Ciphers, and Secret Writing.

In the first months of 2000, I put this paradox (which, afterwards, was called "Kraitchik's paradox", a "name " never used by Maurice Kraitchik!) in several magazines and on several lists. Marc Heremans did the same.

As a result, we got more than 50 answers! Most of them did not answer to anything, or were very poor.

Finally, two articles came, giving finally what I consider to be "The Solution": One from Marc Heremans, and one from Erik Goolaerts. Also, Chris Langan wrote an interesting solution on: http://www.megafoundation.org/Ubiquity/Paradox.html

Here is the solution founded by Marc Heremans:

Paradox, antinomy or sophism, I don't know which term best describes this statement.

Still, it generates the simultaneous feeling of admiration and incredulity, close to the one that one feels when a devious lawyer misleads his public while pleading brilliantly an already lost cause.

We have the conviction of having been fooled, certainly, but the tracks are covered so finely that it is difficult for us to unmask the deception.

The attempts to resolve the paradox, which call on the general laws of logic and simple "common sense", are shown to be useless because they confirm a logical impossibility of which we are perfectly conscious but do not tell us where the error lies.

Let's try to understand why the reasoning is not correct.

A visual representation in the form of a matrix will help.

- *The amounts of player A (a_1, a_2,....,a_n) can be seen in the left column and the amounts of player B (b_1, b_2,...,b_n) are shown in the top row*
- *The gains shown in the cells correspond with player A's point of view (when he wins, he receives the amount of his opponent; when he loses, he only loses his amount)*
- *To simplify the presentation, we will assume that the amounts are in whole units (Euros, dollars, etc.) and*
- *that their distribution is uniform (a binomial distribution seems more realistic in practice, but does not change anything fundamental to the reasoning ; it makes it, merely, technically more difficult)*
- *Let us consider an uneven number of amounts (e.g. 5) in order to have a central value ($a_3=2$ in the present case), the minimum amount (a_1) being equal to 0 ;*
- *The last column shows the total of wins for each occurrence of the variable "a" (in brackets, the mathematical expectation "E").*

	0	1	2	3	4	Total (E)
0	0	+1	+2	+3	+4	10 (2)
1	-1	0	+2	+3	+4	8 (1.60)
2	-2	-2	0	+3	+4	3 (0.6)
3	-3	-3	-3	0	+4	-5 (-1)
4	-4	-4	-4	-4	0	-16 (-3.2)

Remarks about the matrix

Notice (this will become important later) that the matrix is either symmetric or not, depending on the way you look at it.

If one considers the amounts shown on either side of the diagonal made up of the zero wins/losses, one can note that the matrix is perfectly symmetrical, each positive value being matched with an equivalent negative value. All the mathematical expectations are complementary and cancel each other out. In half of the cases, the "game" is favourable for A; in the other half, it is favourable for B. The number of winning positions is the same as the number of losing ones and the losing amounts are equal to the winning ones.

A second approach consists of no longer looking at all the possible occurrences, as above, but to regroup the data, taking into consideration the regression of the "a_i" on the b_i For each occurrence of "a", we associate its average expectation of gain. Therefore, we are interested, in order of priority, in the first (a_1,…,a_n) and the last column (E1,...,E_n).

Seen from this angle, the matrix is no longer symmetrical. The expected values vary greatly from one amount a_i to another. The number of winning positions is even superior to the number of losing ones! On the other hand, the amounts of the losses are greater than the amounts of the gains.

We can note that the mathematical expectation of A is clearly positive when he holds an "average" amount!

The reasoning proposed by A (refer to Albert Frank's previous article): "if I win – probability 0.5 – the final amount in my possession will be greater than 2A," shows itself to be correct for the specific case of an average amount, but can not be generalised.

Actually, the fact of winning on average when you hold an average amount does not at all mean to win "on average" in all possible cases.

That would be to ignore the totally asymmetrical shape of the distribution of wins and losses around the average amount. In extreme situations, the wins and losses are not balanced. A loses much more when he is in possession of a high amount than he would win when he owns a small amount.

Conclusion

A's error consists of reasoning that does not take into account alternative groupings of the data he uses.

He goes from one grouping representation to another, transposing surreptitiously conclusions that could be drawn from the other.

By itself, no grouping is "better" than any other, but if one accepts a reading of the matrix based on the second approach (asymmetrical), it is necessary to take into consideration the unequal values of the expectations resulting from the grouping of the amounts that were used in that line of reasoning.

And it is quite easy to see that Pascal's wager and Kraïtchik's paradox are nearly the same, with a totally similar structure.

Because of this similarity, what we can now call "The Paradox of Pascal" can be solved in the same way that Marc Heremans and Erik Goolaerts solved the Parodox of Kraïtchik. Without knowing they where, they have solved a *very* old paradox.

This is a *very* good example of how a problem can be sometimes solved because there is an isomorphism with another problem that has been solved before."

In the same way that it has been demonstrated, in the Kraïtchik's paradox, that when one of the players says "it is in my favour", he is wrong (and the bet is equilibrated), we can say that Pascal was wrong when he said " you must wager for the existence of god" - He made the same mistake when looking at the expectation - and the bet is equilibrated (you don't loose(or win) more - if you loose - betting against his wager – that you would loose(or win) betting for it).